Critical exponents and corrections to scaling for bond trees in two dimensions

1989 J. Phys. A: Math. Gen. 22 4419

(http://iopscience.iop.org/0305-4470/22/20/018)

J. Phys. A: Math. Gen. 22 (1989) 4419-4431. Printed in the UK

Critical exponents and corrections to scaling for bond trees in two dimensions

T Ishinabe

Faculty of Engineering, Yamagata University, Yonezawa 992, Japan

Received 9 March 1989, in final form 26 May 1989

Abstract. We have analysed the newly obtained series of the radius of gyration R_n and the number of clusters N_n for n-bond trees (i.e. branch polymers without loops) on the square ($n \leq 14$) and triangular ($n \leq 11$) lattices to estimate the critical parameters. Respective estimates of the exponents ν and θ for R_n and N_n are consistent with the corresponding values for lattice animals, while the correction-to-scaling exponent Δ_1 is inconsistent with the animal value. In addition, Δ_1 has different values for R_n ($\Delta_1 = 0.635$) and N_n ($\Delta_1 = 1.3$) for bond trees. We have also estimated an exponent δ characterising the density distribution; $\delta = 2.69$ for the triangular lattice.

1. Introduction

Lattice animals and lattice trees (i.e. lattice animals with no loops) serve as models of random branch polymers in dilute solutions. The statistics of lattice animals is essentially identical to that of percolation clusters below percolation threshold (Family and Coniglio 1980, Harris and Lubensky 1981), and has been extensively investigated theoretically and numerically. However, such attempts for lattice trees are comparatively scarce although renormalisation group (RG) theories (Lubensky and Isaacson 1979, Family 1980, 1982a) predict that they are in the same universality class as animals.

For animals and trees, the mean-square radius of gyration R_n^2 and the number of clusters N_n with n elements (bonds or sites) can be written for large n as

$$R_n^2 = An^{2\nu}(1 + Bn^{-\Delta_1} + \ldots) \tag{1}$$

and

$$N_n = A'n^{-\theta}\lambda^n(1 + B'n^{-\Delta_1} + \ldots). \tag{2}$$

Here ν and θ are leading scaling exponents, and Δ_1 is the correction-to-scaling exponent while λ is the (lattice-dependent) growth constant. Parisi and Sourlas (1981) have found the relations between the exponents (θ and ν) of animals in d dimensions and the exponent σ of the Lee–Yang edge singularity of the Ising model in $d-2$ dimensions (Fisher 1978):

$$\theta(d) = \sigma(d-2) + 2 \tag{3}$$

and

$$\nu(d) = [\sigma(d-2)+1]/(d-2). \tag{4}$$

0305-4470/89/204419+13\$02.50

Substitution of the exact values $\sigma(0)=-1$ and $\sigma(1)=-\frac{1}{2}$ into (3) and (4) yields $\theta(2)=1$, $\theta(3)=\frac{3}{2}$ and $\nu(3)=\frac{1}{2}$ whereas $\nu(2)$ is undetermined. Combining (3) and (4) leads to

$$\nu=(\theta-1)/(d-2). \tag{5}$$

This equation suggests that there is only a single independent exponent for animal problems, as indicated by Family (1982b).

The Flory approximation (Isaacson and Lubensky 1980, Daoud and Joanny 1981) gives

$$\nu=5/[2(d+2)]. \tag{6}$$

Combining (6) with (5), we have

$$\theta=(7d-6)/[2(d+2)]. \tag{7}$$

Equations (6) and (7) reproduce not only the exact values of (3) and (4) for $d=2$ and $d=3$ excepting $\nu(2)$ but also the exact results $\nu(4)=\frac{5}{12}$ and $\theta(4)=\frac{11}{6}$ given by Dhar (1983, 1986) from the exact solution of the hard-square lattice-gas model. For $d=8$, (6) and (7) give the Cayley tree values: $\nu=\frac{1}{4}$ (Zimm and Stockmayer 1949) and $\theta=\frac{5}{2}$ (Fisher and Essam 1961); this suggests that the critical dimension $d_c=8$, as predicted from a field-theoretical calculation (Lubensky and Isaacson 1979) and confirmed from exact enumerations (Gaunt 1980). Recently, Gujrati (1988) has asserted $d_c=4$, and that any critical exponents cannot be defined for a single branch polymer since such a system exhibits a first-order transition.

Equation (6) predicts $\nu=0.625$ for $d=2$. Family (1983) has obtained a value consistent with it from the real space RG approach for bond animals (i.e. weak embeddings) while $\nu=0.649$ for site animals (i.e. strong embeddings). Results from the finite-size scaling renormalisation method for site animals are, however, reconciled with $\nu=0.6408$ (Derrida and DeSeze 1982, Derrida and Stauffer 1985, Kertész 1986). Most Monte Carlo estimates of ν are in the range 0.64–0.65 (Gould and Holl 1981, Djordjevic *et al* 1984, Havlin *et al* 1984, Caracciolo and Glaus 1985, Dhar and Lam 1986) while an exact enumeration (Peters *et al* 1979) gives $\nu=0.65$.

As for N_n, the exact value $\theta=1$ in 2D is confirmed from exact enumerations (Gaunt *et al* 1976, Guttmann and Gaunt 1978, Peters *et al* 1979, Adler *et al* 1988) and a Monte Carlo technique (Caracciolo and Glaus 1985). The first estimation of Δ_1 has been done by Guttmann and Gaunt (1978) from the analysis of exact series data for site and bond animals assuming $\theta=1$; they estimate $\Delta_1\simeq 1$ although a tendency $\Delta_1(\text{bond})>\Delta_1(\text{site})$ is found. Similarly, Guttmann (1982) has obtained, however, $\Delta_1=0.87$ by exploiting the longer series given by Redelmeier (1981). Other methods (Margolina *et al* 1983, Privman 1984, Adler *et al* 1988) using the same data are in favour of this value while Margolina *et al* (1984b) have estimated $\Delta_1=0.75$ from another series analysis.

Privman (1984) has obtained $\Delta_1=0.83$ and $\nu=0.6412$ by applying the method of Adler *et al* (1983) to the finite-size scaling data of Derrida and DeSeze (1982). On the other hand, Margolina *et al* (1984a) have found $\nu=0.640$ by analysing the R_n series of Peters *et al* (1979) with the method of Privman and Fisher (1983) for the choice of Guttmann's value $\Delta_1=0.87$. Assuming this value of ν, Family *et al* (1985) have estimated Δ_1 consistent with Guttmann's value from the analysis of the anisotropy of the radius of gyration tensor; the unbiased estimate without that assumption gives $\Delta_1=1.01$. Lam (1986) has estimated $\Delta_1=0.5$ from R_n series assuming $\nu=0.6406$.

Gaunt *et al* (1982) as well as Duarte and Ruskin (1981) have shown that $\theta = 1$ for bond and site trees in 2D on the basis of exact enumerations; this confirms that both site and bond trees belong to the same universality class as site and bond animals. Seitz and Klein (1981) have estimated $\nu = 0.615$ for trees while the real space RG calculation (Family 1980) gives $\nu = 0.6370$. Recent Monte Carlo approaches (Caracciolo and Glaus 1985, Meirovitch 1987) estimate $\nu = 0.640$ and $\theta = 1.00$ for bond trees by assuming $\Delta_1 = 1$; these estimates are in accord with the animal values.

The density distribution $P_n(\boldsymbol{r})$ at a point $\boldsymbol{r}$ from the centre of mass of n-clusters can be described using an exponent δ as

$$P_n(\boldsymbol{r}) \simeq A \exp[a(r/R_n)^\delta] \tag{8}$$

for sufficiently large n (Stauffer 1978a). Herrmann (1979) has assessed $\delta = 2.6$ for animals in 2D by use of the Monte Carlo data given by Peters *et al* (1979).

In this paper, we estimate the values of λ, θ, ν, Δ_1 and δ in 2D using the exact series data newly obtained for bond trees (i.e. lattice animals without loops and weakly embeddable in the lattice) on the square (SQ) and triangular (TRI) lattices; much attention is paid to ν, Δ_1 and δ. To our knowledge, the estimation of Δ_1 and δ is the first attempt for lattice trees although the analysis of the anisotropy of the radius of gyration tensor by Family *et al* (1985) suggests that Δ_1 is equivalent for animal and tree.

2. Exact enumerations

We have carried out exact enumerations of n-bond trees on the TRI and SQ lattices for up to $n = 11$ and 14, respectively, by exploiting the Martin algorithm (Martin 1974, Redner 1982). We have added three more terms to both the existing series of N_n for the TRI (Duarte and Ruskin 1981) and SQ (Gaunt *et al* 1982) lattices while the R_n series were new; these series are reproduced in table 1. The series for $P_n(\boldsymbol{r})(n \leq 11)$ for the TRI lattice were enumerated in the form of $N_n P_n(q_1, q_2)$ using the oblique coordinate system, where $r^2 = q_1^2 + q_1 q_2 + q_2^2$ (see Ishinabe 1987). For the SQ lattice

Table 1. Exact series of N_n and R_n for the SQ and TRI lattices.

	SQ		TRI	
n	N_n	$(n+1)^2 N_n R_n^2$	N_n	$(n+1)^2 N_n R_n^2$
1	2	2	3	3
1	6	28	15	66
3	22	276	89	1 050
4	87	2 320	576	14 334
5	364	17 780	3 930	178 578
6	1 574	127 844	27 782	2 092 416
7	6 986	879 036	201 414	23 454 906
8	31 581	5 839 760	1 488 048	254 233 146
9	144 880	37 772 428	11 156 061	2 683 896 297
10	672 390	239 082 260	84 622 074	27 735 573 846
11	3 150 362	1 486 548 912	648 039 990	281 619 667 638
12	14 877 317	9 105 610 904		
13	70 726 936	55 068 644 440		
14	338 158 676	329 401 857 232		

only the series $N_nP_n(x)(n \leqslant 14)$ of the x-component distribution were obtained, owing to our computer memory. These P_n series are not given here since the tables are too lengthy to reproduce, but they are available upon request.

3. Series analysis

3.1. Radius of gyration

We estimate ν and Δ_1 following a method (Ishinabe 1988, 1989) based on the conventional technique of series analysis combined with the finite-size scaling idea of Privman and Fisher (1983), employing the cancellation of leading correction terms. First we evaluate the ratios†

$$\nu_{n,k} = \tfrac{1}{2} n(\rho_{n+k}/\rho_n - 1)/k \tag{9}$$

for $k = 1$ or 2, where $\rho_n \equiv R_n^2$. The ratios ($k = 1$) of adjacent terms are used for the TRI lattice while the alternate ratios ($k = 2$) are used for the SQ lattice. After forming these ratios, we construct the Neville table (e.g. Gaunt and Guttmann 1974) for linear, quadratic and cubic extrapolants

$$\nu_{n,k}^{(r)} = [n\nu_{n,k}^{(r-1)} - (n - kr)\nu_{n-k,k}^{(r-1)}]/kr \tag{10}$$

for $r = 1$-3, with $\nu_{n,k}^{(0)} \equiv \nu_{n,k}$. We determine the first trial value of ν by plotting these extrapolants against n^{-1} and extrapolate to $n \to \infty$, having in mind the curvature of convergence as a whole together with damping oscillations. Then the estimators

$$B_{n,k}(\Delta_1) = \frac{\rho_n(n-k)^{2\nu}\rho_{n-k}n^{2\nu}}{\rho_{n-k}n^{2\nu-\Delta_1} - \rho_n(n-k)^{2\nu-\Delta_1}} \tag{11}$$

are constructed. The curves $B_{n,k}(\Delta_1)$ as a function of Δ_1 for different n intersect at a point close to the correct Δ_1 if ν is known; approximate values of Δ_1 and B can be estimated for the trial ν. We perform the transformation $\rho_n^* = \rho_n/(1 + Bn^{-\Delta_1})$, using the result to eliminate the singular term. Similarly, the improved ν is estimated from ρ_n^* series. Thus we get reliable estimates of ν and Δ_1 by repeating the above procedure several times. We can also estimate Δ_1 and A from the estimators (Privman 1984)

$$A_{n,k}(\Delta_1) = \frac{\rho_n n^{\Delta_1 - 2\nu} - \rho_{n-k}(n-k)^{\Delta_1 - 2\nu}}{n^{\Delta_1} - (n-k)^{\Delta_1}}. \tag{12}$$

Figure 1 illustrates the first plots of $\nu_{n,2}^{(r)}$ ($r = 1$ and 2) against n^{-1} for the SQ lattice; we get $\nu = 0.640 \pm 0.008$ as a trial value. Using this value, we have $\Delta_1 = 0.67$ and $B = 1.24$ from the intersection of $B_{n,2}(\Delta_1)$ curves for different n. An improved estimate of ν is obtained by exploiting the transformed series ρ_n^*; some terms in the appropriate Neville table of $\nu_{n,2}^{(r)}$ ($r = 1$-3) for ρ_n^* are listed in table 2. We take $\nu = 0.644 \pm 0.004$ as our final estimate in view of the increase in the last five terms of $\nu_{n,2}^{(1)}$ as n increases, but with a tendency to be somewhat bowed downwards, and the upwards trend as a whole with a slight odd-even oscillation in those of $\nu_{n,2}^{(2)}$. We show $B_{n,2}(\Delta_1)$ curves for $n = 10$–14 for $\nu = 0.644$ in figure 2; the successive average $\bar{B}_{n,2} = \tfrac{1}{2}(B_{n-1,2} + B_{n,2})$ is employed in place of $B_{n,2}$ to lessen the odd-even effect for the SQ lattice, but we omit the bar in

† We also used the ratios $\nu_{n,k} = \tfrac{1}{2}\ln(\rho_n/\rho_{n-k})/\ln[n/(n-k)]$, but they yield almost the same values as are obtained from (9).

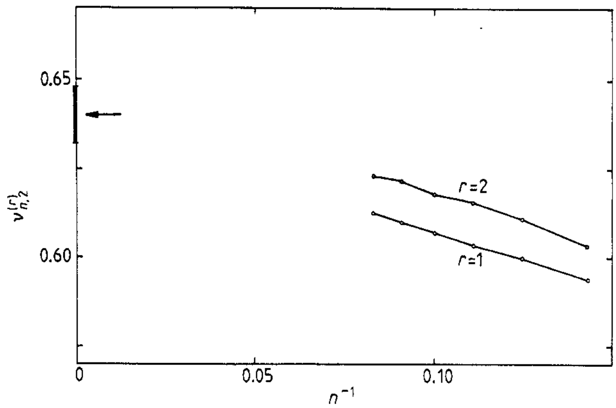

Figure 1. Ratio estimate of ν from linear ($r=1$) and quadratic ($r=2$) extrapolants $\nu_{n,2}^{(r)}$ for the SQ lattice.

Table 2. Neville table for the estimation of ν from transformed R_n series for the SQ lattice.

n	$\nu_{n,2}$	$\nu_{n,2}^{(1)}$	$\nu_{n,2}^{(2)}$	$\nu_{n,2}^{(3)}$
6	0.669 207	0.635 91	0.632 24	
7	0.664 891	0.635 57	0.636 70	0.610 72
8	0.661 461	0.638 22	0.640 53	0.643 29
9	0.659 009	0.638 42	0.641 98	0.644 62
10	0.657 113	0.639 72	0.641 97	0.642 93
11	0.655 629	0.640 42	0.643 91	0.645 51
12	0.654 445	0.641 11	0.643 88	0.645 80

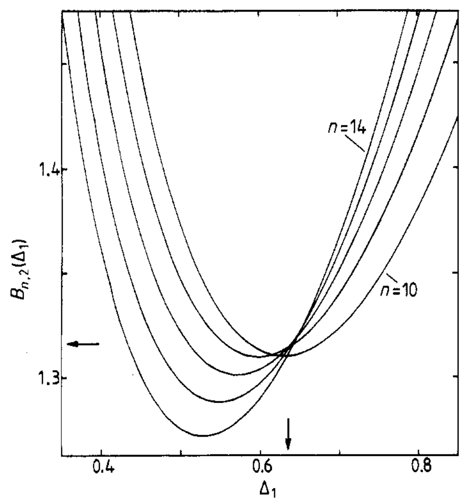

Figure 2. Curves of $B_{n,2}(\Delta_1)$ for the input $\nu=0.644$ for the SQ lattice.

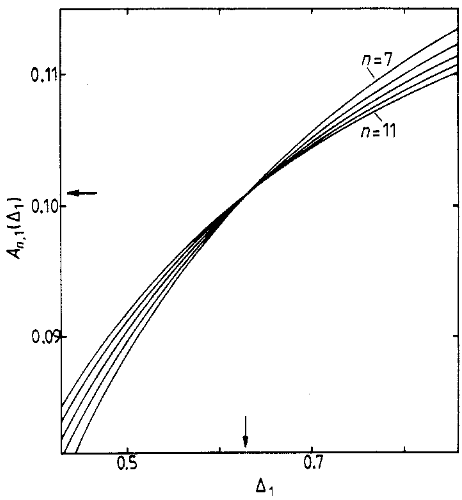

Figure 3. Curves of $A_{n,1}(\Delta_1)$ for the imput $\nu=0.648$ for the TRI lattice.

$\bar{B}_{n,2}$. We get $\Delta_1 = 0.635$ and $B = 1.318$; almost the same value of Δ_1 is estimated from the corresponding $A_{n,2}(\Delta_1)$ curves. We take $\Delta_1 = 0.635 \pm 0.03$ as our final estimate; the error limit is determined by considering that of ν. Similarly, we estimate $\nu = 0.648 \pm 0.005$ for the TRI lattice; some terms in the corresponding Neville table of $\nu_{n,1}^{(r)}$ ($r = 1$-3) for ρ_n^* series are reproduced in table 3. The $A_{n,1}(\Delta_1)$ curves for $n = 7$-11 for $\nu = 0.648$ are depicted in figure 3; we get $\Delta_1 = 0.635 \pm 0.02$ and $A = 0.1156$. The intersection of such curves for $B_{n,1}(\Delta_1)$ gives the same value of Δ_1 and $B = 1.528$.

3.2. Number of trees

For N_n series we form

$$\lambda_{n,k} = (N_n / N_{n-k})^{1/k} \tag{13}$$

and

$$\theta_{n,k}^{(r)} = -n(\lambda_{n,k}/\lambda_{n,k}^{(r)} - 1) \tag{14}$$

where $\lambda_{n,k}^{(r)}$ ($r = 1$-3) are the rth extrapolants of $\lambda_{n,k}$ defined as in (10), and $k = 1$ (TRI) and 2 (SQ). The plots of $\theta_{n,2}^{(r)}$ against n^{-1} for $r = 1$ and 2 are shown in figure 4 for the SQ lattice; they suggest $\theta = 1.01 \pm 0.015$ while we have $\theta = 1.01 \pm 0.02$ from the similar plots of $\theta_{n,1}^{(r)}$ for the TRI lattice. Our results are compatible with the other estimates for lattice trees (Duarte and Ruskin 1981, Gaunt *et al* 1982) and in good agreement

Table 3. Neville table for the estimation of ν from transformed R_n series for the TRI lattice.

n	$\nu_{n,1}$	$\nu_{n,1}^{(1)}$	$\nu_{n,1}^{(2)}$	$\nu_{n,1}^{(3)}$
4	0.666 652	0.637 45	0.634 59	0.633 57
5	0.660 656	0.636 67	0.635 51	0.636 13
6	0.657 066	0.639 11	0.643 99	0.652 47
7	0.654 665	0.640 26	0.643 13	0.641 99
8	0.652 980	0.641 19	0.643 96	0.645 34
9	0.651 789	0.642 26	0.646 02	0.650 15
10	0.650 920	0.643 09	0.646 41	0.647 33

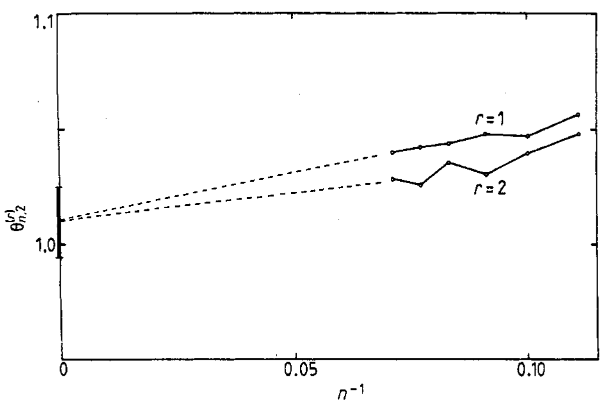

Figure 4. Ratio estimate of θ from linear ($r = 1$) and quadratic ($r = 2$) extrapolants $\theta_{n,2}^{(r)}$ for the SQ lattice.

with the commonly accepted value $\theta = 1$ for lattice animals in 2D. The analogous plots for $\lambda^{(r)}_{n,k}$ produce $\lambda = 5.140 \pm 0.008$ and $\lambda = 8.41 \pm 0.02$ for the SQ and TRI lattices, respectively. These values are compared with the corresponding estimates $\lambda = 5.14 \pm 0.01$ (Gaunt *et al* 1982) and $\lambda = 8.40 \pm 0.03$ (Duarte and Ruskin 1981).

If λ and θ are evaluated, we can estimate Δ_1 from a method similar to that for R_n by forming

$$A'_{n,k}(\Delta_1) = \frac{N_n n^{\Delta_1+\theta} - N_{n-k}\lambda^k (n-k)^{\Delta_1+\theta}}{\lambda^n[n^{\Delta_1} - (n-k)^{\Delta_1}]} \tag{15}$$

or

$$B'_{n,k}(\Delta_1) = \frac{N_n(n-k)^{-\theta} - N_{n-k}\lambda^k n^{-\theta}}{N_{n-k}\lambda^k n^{-\theta-\Delta_1} - N_n(n-k)^{-\theta-\Delta_1}}. \tag{16}$$

It is practically difficult, however, to reliably estimate Δ_1 in this case since the confidence of the estimate has a strong dependence on error limits of both λ and θ. We then evaluate

$$\lambda_{n,k} = [nN_n/(n-k)N_{n-k}]^{1/k} \tag{13'}$$

instead of (13) assuming $\theta = 1$, and estimate Δ_1 and B' by exploiting (16) with $\theta = 1$. A reliable value of λ is obtained using the transformation $N^*_n = N_n/(1 + B'n^{-\Delta_1})$.

Figure 5 shows $B'_{n,2}(\Delta_1)$ curves ($n = 11$–14) for the SQ lattice for $\lambda = 5.143$ obtained from (13′); the intersection of the curves yields $B' = -0.376$ and $\Delta_1 = 1.34 \pm 0.50$, where the error limit is estimated by considering that of λ. The corresponding estimate for $A'_{n,2}(\Delta_1)$ gives $A' = 0.527$ and the same Δ_1. The improved value $\lambda = 5.142 \pm 0.002$ is obtained for the SQ lattice from the transformed N^*_n series. Similarly, we have $\Delta_1 = 1.26 \pm 0.50$, $A' = 0.4884$, $B' = -0.312$ and $\lambda = 8.412 \pm 0.004$ for the TRI lattice. We reproduce some terms in the Neville tables of $\lambda^{(r)}_{n,2}$ and $\lambda^{(r)}_{n,1}$ ($r = 1$–3) for N^*_n series in tables 4 and 5 for the SQ and TRI lattices, respectively.

3.3. *Density distribution*

In figure 6 we show the cross sections of $P_n(\boldsymbol{r})$ in different directions as a function of r/R_n; they were obtained from our enumeration data for $n = 11$ for the TRI lattice.

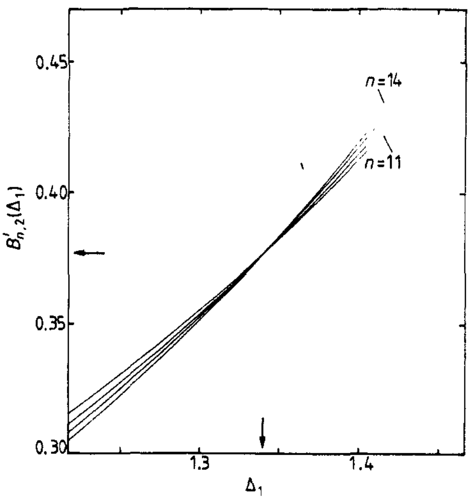

Figure 5. Curves of $B'_{n,2}(\Delta_1)$ for the inputs $\lambda = 5.143$ and $\theta = 1$ for the SQ lattice.

Table 4. Neville table for the estimation of λ from transformed N_n series for the SQ lattice.

n	$\lambda_{n,2}$	$\lambda_{n,2}^{(1)}$	$\lambda_{n,2}^{(2)}$	$\lambda_{n,2}^{(3)}$
8	5.143 315	5.145 28	5.105 97	5.922 89
9	5.142 797	5.148 07	5.129 21	5.317 51
10	5.143 103	5.142 26	5.137 72	5.158 89
11	5.143 038	5.144 12	5.137 21	5.143 88
12	5.143 051	5.142 79	5.143 86	5.150 00
13	5.143 037	5.143 03	5.140 58	5.144 50
14	5.143 030	5.142 91	5.143 21	5.142 34

Table 5. Neville table for the estimation of λ from transformed N_n series for the TRI lattice.

n	$\lambda_{n,1}$	$\lambda_{n,1}^{(1)}$	$\lambda_{n,1}^{(2)}$	$\lambda_{n,1}^{(3)}$
6	8.408 297	8.405 66	8.411 25	8.509 09
7	8.407 414	8.402 12	8.393 26	8.369 27
8	8.407 016	8.404 23	8.410 57	8.439 42
9	8.406 937	8.406 31	8.413 56	8.419 55
10	8.406 945	8.407 01	8.409 85	8.401 17
11	8.407 016	8.407 73	8.410 94	8.413 87

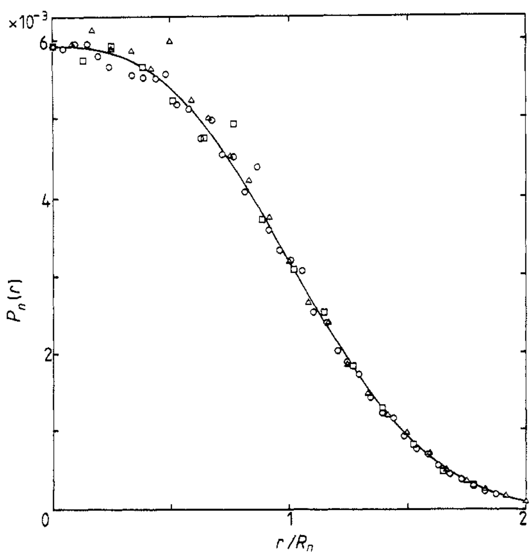

Figure 6. Cross sections of $P_n(r)$ ($n=11$) for the TRI lattice in different directions: $x(=q_1)$ axis (○), 19.1° from the axis (△) and 30° from the x axis (□). The full curve represents (8) for $\delta=2.69$.

These plots suggest that the circular symmetry of $P_n(\boldsymbol{r})$ is maintained in this case and $P_n(\boldsymbol{r})$ can be described by a single exponent δ although the plots are somewhat dispersed for $r/R_n < 1$; such dispersion is also noticed in Monte Carlo data for lattice animals (Stauffer 1978b).

Reduced radial moments are defined by

$$m_{2k}^{(n)} = \langle r_n^{2k} \rangle / \langle r_n^2 \rangle^k \tag{17}$$

where the mean values $\langle r_n^{2k} \rangle$ of $2k$th power of r are evaluated from

$$\langle r_n^{2k} \rangle = \sum_{\boldsymbol{r}} r^{2k} P_n(\boldsymbol{r})/(n+1). \tag{18}$$

The reduced moments of function (8) can be expressed in terms of gamma functions (McKenzie 1973)

$$m_{2k} = \frac{\Gamma((d+2k)/\delta)}{\Gamma(d/\delta)} \left(\frac{\Gamma(d/\delta)}{\Gamma((d+2)/\delta)} \right)^k. \tag{19}$$

We calculate $m_{2k}^{(n)}$ for $k = 2$–6 for the TRI lattice to estimate the limiting values of $m_{2k}^{(\infty)}$ by plotting the extrapolants $m_{n,1}^{(r)}$ ($r = 1$ and 2) against n^{-1}. An example ($k = 2$) of the estimation is depicted in figure 7; we obtain $m_4^{(\infty)} = 1.765 \pm 0.008$, having in mind the monotonic trend of the plots. The values of $m_{2k}^{(\infty)}$ thus estimated are listed in table 6 together with those of $m_{2k}^{(n)}$ for $n \leq 11$. We compare the $m_{2k}^{(\infty)}$ values with the table of m_{2k} evaluated from (19) for each k for the appropriate δ to find a region of δ such that each value of $m_{2k}^{(\infty)}$ ($k = 2$–6) is contained in the corresponding region of m_{2k}. Thus we have

$$\delta = 2.69^{+0.11}_{-0.10} \tag{20}$$

where the error limit is taken in view of the estimated errors in $m_{2k}^{(\infty)}$; our value is compatible with the value $\delta = 2.6$ for a lattice animal given by Herrmann (1979). The m_{2k} values calculated from (19) for $\delta = 2.69$ are also given in table 6 (in parentheses) for the sake of comparison. The full curve in figure 6 represents $P_n(\boldsymbol{r})$ which is best-fitted to the data of $n = 11$, i.e. $A = 0.005\,92$, and $a = 0.6309$ with $\delta = 2.69$.

For the SQ lattice, we consider the x component quantities

$$M_{2k}^{(n)} = \langle x_n^{2k} \rangle / \langle x_n^2 \rangle^k \tag{21}$$

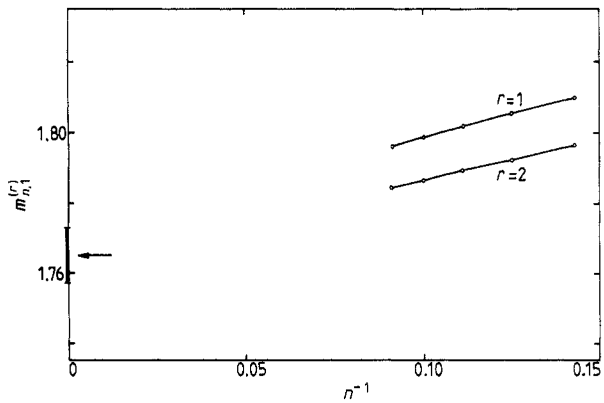

Figure 7. Plots of linear and quadratic extrapolants $m_{n,1}^{(r)}$ of $m_4^{(n)}$ against n^{-1} for the TRI lattice.

Table 6. Values of $m_{2k}^{(n)}$ ($k=1$-6) and $m_{2k}^{(\infty)}$ for the TRI lattice.

n	$m_4^{(n)}$	$m_6^{(n)}$	$m_8^{(n)}$	$m_{10}^{(n)}$	$m_{12}^{(n)}$
1	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
2	1.425 620	2.343 163	4.129 030	7.550 743	14.128 882
3	1.563 978	3.042 198	6.699 744	15.927 908	39.830 034
4	1.630 428	3.428 300	8.438 464	23.048 611	67.660 481
5	1.668 653	3.664 857	9.612 789	28.511 521	92.396 121
6	1.693 106	3.821 168	10.431 771	32.617 885	112.792 427
7	1.709 803	3.929 824	11.020 040	35.710 925	129 122 067
8	1.721 775	4.008 538	11.455 069	38.070 877	142.110 532
9	1.730 685	4.067 444	11.784 958	39.898 950	152.470 558
10	1.737 505	4.112 652	12.040 321	41.335 274	160.784 904
11	1.742 845	4.148 074	12.241 507	42.478 851	167.509 008
∞	1.765 ± 0.008	4.29 ± 0.05	13.0 ± 0.5	46.5 ± 3.5	197 ± 28
	(1.7622)	(4.278)	(12.99)	(46.86)	(194.3)†

† The figures in parentheses are the values of m_{2k} calculated from (19) for $\delta = 2.69$.

and

$$M_{2k} = \frac{\Gamma((2k+1)/\delta)}{\Gamma(1/\delta)} \left(\frac{\Gamma(1/\delta)}{\Gamma(3/\delta)} \right)^k \tag{22}$$

since we only have the series of x component distribution $P_n(x) \sim \exp[a'(x/R_n)^\delta]$ (Domb *et al* 1965). Following the method mentioned above, we get the estimate $\delta = 2.54 \pm 0.12$, which is somewhat smaller than (20).

We list the values of $M_{2k}^{(n)} (n \leqslant 14)$, $M_{2k}^{(\infty)}$, and M_{2k} for $\delta = 2.54$ in table 7 for $k = 2$-6. The corresponding value of δ for the TRI lattice is $\delta = 2.52 \pm 0.10$.

Table 7. Values of $M_{2k}^{(n)}$ ($k=1$-6) and $M_{2k}^{(\infty)}$ for the SQ lattice.

n	$M_4^{(n)}$	$M_6^{(n)}$	$M_8^{(n)}$	$M_{10}^{(n)}$	$M_{12}^{(n)}$
1	2.000 000	4.000 00	8.000 00	16.000 00	32.000 00
2	2.387 755	7.530 61	26.527 70	98.157 14	371.533 94
3	2.516 068	9.230 00	40.696 60	198.029 59	1020.742 93
4	2.574 828	9.919 40	47.958 17	266.465 04	1617.248 74
5	2.607 107	10.322 23	52.443 77	313.479 39	2092.887 68
6	2.626 568	10.554 30	55.106 06	343.708 20	2432.119 96
7	2.639 375	10.706 05	56.857 67	364.299 10	2677.185 71
8	2.648 074	10.805 48	58.001 01	378.095 35	2848.923 77
9	2.654 219	10.873 73	58.778 63	387.604 36	2970.848 94
10	2.658 677	10.921 58	59.315 00	394.206 81	3057.417 59
11	2.661 964	10.955 54	59.688 54	398.821 09	3119.032 43
12	2.664 423	10.979 89	59.949 83	402.049 36	3162.834 98
13	2.666 279	10.997 37	60.131 75	404.293 44	3193.781 61
14	2.667 684	11.009 82	60.256 29	405.825 14	3215.314 68
∞	2.650 ± 0.008	10.7 ± 0.1	56.2 ± 1.2	350 ± 14	2400 ± 200
	(2.6100)	(10.434)	(54.96)	(355.1)	(2698)†

† The figures in parentheses are the values of M_{2k} calculated from (22) for $\delta = 2.54$.

4. Discussion and conclusion

Critical parameters estimated for bond trees on the SQ and TRI lattices are listed in table 8. The values of ν are reconciled with most Monte Carlo results $\nu = 0.64$–0.65 for lattice animals and $\nu = 0.649 \pm 0.009$ from the real space RG theory (Family 1983), while not consistent with the Flory value of $\frac{5}{8}$. Recent Monte Carlo estimations (Caracciolo and Glaus 1985, Meirovitch 1987) and the finite-size scaling renormalisation method (Derrida and Stauffer 1985) give slightly smaller values $\nu = 0.640 \pm 0.008$ and $\nu = 0.640\,75 \pm 0.000\,15$ for bond trees and site animals, respectively; they are almost on the limits of the estimated uncertainties in our estimates.

Table 8. Critical values estimated from (*a*) R_n and (*b*) N_n series for the SQ and TRI lattices.

(*a*)

Lattices	ν	Δ_1	A	B
SQ	0.644 ± 0.004	0.635 ± 0.03	0.1156	1.318
TRI	0.648 ± 0.005	0.635 ± 0.02	0.1021	1.528

(*b*)†

	λ	Δ_1	A'	B'
SQ	5.142 ± 0.002	1.34 ± 0.5	0.527	-0.376
TRI	8.412 ± 0.004	1.26 ± 0.5	0.488	-0.312

† $\theta = 1$ is assumed.

Our estimation of Δ_1 from R_n series yields a smaller value than the commonly accepted value $\Delta_1 = 0.87 \pm 0.06$ for lattice animals given by Guttmann (1982) while that from N_n series yields a significantly larger value. Our estimation of Δ_1 is dependent on a given value of ν or λ. The dependence is, however, very weak for R_n; we have $\Delta_1 = 0.67$ and 0.66 for the SQ and TRI lattices, respectively, if $\nu = 0.640$ is adopted. We have checked our method using the existing series of $N_n\,(n \leqslant 24)$ obtained by Redelmeier (1981) for a site animal on the SQ lattice. We have $\Delta_1 = 0.83 \pm 0.06$, $A' = 0.3174$ and $B' = -0.412$ assuming $\theta = 1$ and $\lambda = 4.0626 \pm 0.0002$ (Guttmann 1982); the value of Δ_1 is consistent with the accepted value, and A' and B' are compared with Guttmann's $A = 0.317$ and $B' = -0.465$. We also obtain $\Delta_1 = 0.78 \pm 0.05$, $A = 0.1897$, $B = -0.748$ and $\nu = 0.641 \pm 0.002$ by applying our method to the R_n series ($n \leqslant 19$) of Lam (1986); they are reconciled with accepted values for lattice animals. Hence, our results suggest a possibility that Δ_1 is different not only between trees and animals but also between R_n and N_n for lattice trees although it is not very conclusive in the latter case due to large error limit of Δ_1 for N_n.

We have tried another approach to estimate Δ_1 for N_n following Privman (1984). Putting $\theta = 1$ we form

$$\lambda_{n,k}(\Delta_1) = \left(\frac{\lambda_{n,k} n^{\Delta_1+1} - \lambda_{n-k,k}(n-k)^{\Delta_1+1}}{n^{\Delta_1+1} - (n-k)^{\Delta_1+1}} \right)^{1/k} \tag{23}$$

for the TRI ($k = 1$) and SQ ($k = 2$) lattices, where $\lambda_{n,k}$ is evaluated from (13′). The values of λ and Δ_1 are estimated simultaneously from the intersection of $\lambda_{n,k}(\Delta_1)$ curves for different n. The results are $\lambda = 8.409$ and $\Delta_1 = 1.02$ for the TRI lattice while $\lambda = 5.144$ and $\Delta_1 = 1.73$ for the SQ lattice. The marked discrepancy in Δ_1 between two lattices suggests the error limit of Δ_1 is very large also in this unbiased method. It seems that these results are not in contradiction with our estimates of Δ_1 in view of the uncertainties of estimates.

In conclusion, we have obtained new series for R_n and $P_n(\boldsymbol{r})$ for bond trees on the TRI and SQ lattices while some new terms are added to the extant data of N_n. Respective estimates of leading scaling exponents ν, θ and δ from R_n, N_n and $P_n(\boldsymbol{r})$ series are in accord with those for lattice animals to confirm the hypothesis that trees are in the same university class as animals. Estimates of the correction-to-scaling exponent Δ_1 are different from the accepted value for animals. In addition, they suggest that Δ_1 is different between R_n and N_n for lattice trees.

Acknowledgments

I am grateful to Y Arai for assistance with computational work, and also the referee for comments. This work was partly supported by a Grant-in-Aid from the Ministry of Education, Science, and Culture, Japan (No. 63303010).

References

Adler J, Meir Y, Harris A B, Aharony A and Duarté J A M S 1988 *Phys. Rev.* B **38** 4941
Adler J, Moshe M and Privman V 1983 *Percolation Structures and Processes* ed G Deutscher, R Zallen and J Adler (Bristol: Adam Hilger)
Caracciolo S and Glaus U 1985 *J. Stat. Phys.* **41** 95
Daoud M and Joanny J F 1981 *J. Physique* **42** 1359
Derrida B and DeSeze L 1982 *J. Physique* **43** 475
Derrida B and Stauffer D 1985 *J. Physique* **46** 1623
Dhar D 1983 *Phys. Rev. Lett.* **51** 853
—— 1986 *Physica* **140A** 210
Dhar D and Lam P M 1986 *J. Phys. A: Math. Gen.* **19** L1057
Djordjevic Z V, Havlin S, Stanley H E and Weiss G H 1984 *Phys. Rev.* B **30** 478
Domb C, Gillis J and Wilmers G 1965 *Proc. Phys. Soc.* **85** 625
Duarte J A M S and Ruskin H J 1981 *J. Physique* **42** 1585
Family F 1980 *J. Phys. A: Math. Gen.* **13** L325
—— 1982a *Phys. Lett.* **92A** 341
—— 1982b *J. Phys. A: Math. Gen.* **15** L583
—— 1983 *J. Phys. A: Math. Gen.* **16** L97
Family F and Coniglio A 1980 *J. Phys. A: Math. Gen.* **13** L403
Family F, Vicsek T and Meakin P 1985 *Phys. Rev. Lett.* **55** 641
Fisher M E 1978 *Phys. Rev. Lett.* **40** 1610
Fisher M E and Essam J W 1961 *J. Math. Phys.* **2** 609
Gaunt D S 1980 *J. Phys. A: Math. Gen.* **13** L97
Gaunt D S and Guttmann A J 1974 *Phase Transitions and Critical Phenomena* vol 3, ed C Domb and M S Green (New York: Academic)
Gaunt D S, Sykes M F and Ruskin H 1976 *J. Phys. A: Math. Gen.* **9** 1899
Gaunt D S, Sykes M F, Torrie G M and Whittington S G 1982 *J. Phys. A: Math. Gen.* **15** 3209
Gould H and Holl K 1981 *J. Phys. A: Math. Gen.* **14** L443
Gujrati P D 1988 *Phys. Rev.* A **38** 5840
Guttmann A J 1982 *J. Phys. A: Math. Gen.* **15** 1987
Guttmann A J and Gaunt D S 1978 *J. Phys. A: Math. Gen.* **11** 949
Harris A B and Lubensky T C 1981 *Phys. Rev.* B **23** 3591
Havlin S, Djordjevic Z V, Majid I, Stanley H E and Weiss G H 1984 *Phys. Rev. Lett.* **53** 178
Herrmann H J 1979 *Z. Phys.* B **32** 335

Isaacson J and Lubensky T C 1980 *J. Physique Lett.* **41** 469
Ishinabe T 1987 *J. Chem. Phys.* **86** 6366
—— 1988 *Phys. Rev.* B **37** 2376
—— 1989 *Phys. Rev.* B **39** 9468
Kertész J 1986 *J. Phys. A: Math. Gen.* **19** 599
Lam P M 1986 *Z. Phys.* B **64** 227
Lubensky T C and Isaacson J 1979 *Phys. Rev.* A **20** 2130
Margolina A, Djordjevic Z V, Stauffer D and Stanley H E 1983 *Phys. Rev.* B **28** 1652
Margolina A, Family F and Privman V 1984a *Z. Phys.* B **54** 321
Margolina A, Nakanishi H, Stauffer D and Stanley H E 1984b *J. Phys. A: Math. Gen.* **17** 1683
Martin J L 1974 *Phase Transitions and Critical Phenomena* vol 3, ed C Domb and M S Green (New York: Academic)
McKenzie D S 1973 *J. Phys. A: Math. Gen.* **6** 338
Meirovitch H 1987 *J. Phys. A: Math. Gen.* **20** 6059
Parisi G and Sourlas N 1981 *Phys. Rev. Lett.* **46** 871
Peters H P, Stauffer D, Hölters H P and Loewenich K 1979 *Z. Phys.* B **34** 399
Privman V 1984 *Physica* **123A** 428
Privman V and Fisher M E 1983 *J. Phys. A: Math. Gen.* **16** L295
Redelmeier D H 1981 *Discrete Math.* **36** 191
Redner S 1982 *J. Stat. Phys.* **29** 309
Seitz W A and Klein D J 1981 *J. Chem. Phys.* **75** 5190
Stauffer D 1978a *Z. Phys.* B **30** 173
—— 1978b *Phys. Rev. Lett.* **41** 1333
Zimm B H and Stockmayer W H 1949 *J. Chem. Phys.* **17** 1301